Bianchi type II, III and V
Diagonal Einstein metrics re-visited

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Abstract

We present, for both minkowskian and euclidean signatures, short derivations of the diagonal Einstein metrics for Bianchi type II, III and V. For the first two cases we show the integrability of the geodesic flow while for the third case a somewhat unusual bifurcation phenomenon takes place: for minkowskian signature elliptic functions are essential in the metric while for euclidean signature only elementary functions appear.

1 Introduction

Modern cosmology [23] has led to a strong development of models based on Bianchi cohomogeneity one metrics. A large amount of information was gathered, mainly for Ricci-flat Bianchi type A metrics (see [17]) either in minkowskian or in euclidean signature. However the need for a cosmological constant leads to consider rather Einstein metrics and not just Ricci-flat ones, leading to more difficult problems.

For the minkowskian signature a complete list of the algebraically special and hypersurface-homogeneous Einstein metrics, using spinors, is given in [12] and many others appear in [13]. For the euclidean signature the most impressive progresses came from Weyl tensor self-duality and culminated with the tri-axial Bianchi type IX self-dual Einstein metrics of Tod and Hitchin [19], [10]. However these ideas give only limited results for the type B metrics as observed in [20]. Another difficulty linked to the type B studies is that even for Ricci-flat geometries there is no need for the metric to be diagonal in the invariant one-forms. Despite these difficulties, quite recently new results were derived for type III [6], type V [5] and type VII_h [18] for which the most general vacuum minkowskian metrics, i. e. non-diagonal ones, were derived.

Our aim is to give very simple derivations of some Einstein metrics, for both euclidean and minkowskian signatures, under the simplifying hypothesis that the metric is *diagonal* with respect to the invariant spatial one-forms. These metrics are examined for Bianchi metrics of type II, III and V. As we shall see they both exhibit interesting features: the types II and III have an integrable geodesic flow and the type V presents an interesting "bifurcation" between the minkowskian and the euclidean regime.

As pointed out by a Referee, the basic integrability of the Einstein equations in the cases considered in this article, which results in our work from an appropriate fixing of the time coordinate, is best understood in a unified and systematic approach if one uses the hamiltonian formalism developed by Uggla et al in [21]. They have shown that the existence of Killing tensors is a key tool leading to a sytematic display of the cases leading to integrability, even if one considers matter and not merely a cosmological constant. This approach is in some sense reminiscent of Carter's derivation of Kerr metric by imposing that it must have a Killing tensor [4].

The content of this article is the following: in Section 2 we present background informations and the field equations for the bi-axial Bianchi type II metrics (they have one extra Killing vector). The metrics are then constructed and, being of type D, their geodesic flow is shown to be integrable.

In Section 3 we present the corresponding construction for the type III metrics. They all share one extra Killing vector. All these metrics are of type D with an integrable geodesic flow. However, in some special cases, there is a strong symmetry enhancment leading to de Sitter, anti de Sitter and $\mathbb{H}^4$ in somewhat unusual coordinates

In section 4 we present the corresponding construction for the type V metrics. Special cases include again de Sitter, anti de Sitter and $\mathbb{H}^4$. However in general one needs elliptic functions to express the minkowskian metrics whereas for the euclidean ones only elementary functions appear.

We give in Appendix A more details on the curious forms of de Sitter metric encountered in the analysis of the Bianchi type III and V Einstein metrics, in Appendix B some technicalities related to elliptic functions and in Appendix C some checks involving curvature computations.

2 Type II metrics

The Bianchi type II Lie algebra is defined as

$$[\mathcal{L}_1, \mathcal{L}_2] = 0, \qquad [\mathcal{L}_2, \mathcal{L}_3] = \mathcal{L}_1, \qquad [\mathcal{L}_3, \mathcal{L}_1] = 0. \tag{1}$$

One can choose "spatial" coordinates (x, y, z) such that

$$\mathcal{L}_1 = \partial_x, \qquad \mathcal{L}_2 = \partial_y - z\,\partial_x, \qquad \mathcal{L}_3 = \partial_z, \tag{2}$$

and the invariant 1-forms

$$\sigma_1 = dx + ydz, \qquad \sigma_2 = dy, \qquad \sigma_3 = dz. \tag{3}$$

We will look for diagonal metrics of the form

$$g = \beta^2\,\sigma_1^2 + \gamma^2(\sigma_2^2 + \sigma_3^2) + \epsilon\,\alpha^2\,dt^2. \tag{4}$$

The bi-axial character of this metric gives a fourth Killing vector

$$\mathcal{L}_4 = y\,\partial_z - z\,\partial_y - \frac{1}{2}(y^2 - z^2)\partial_x, \tag{5}$$

and the algebra closes according to

$$[\mathcal{L}_4, \mathcal{L}_2] = -\mathcal{L}_3, \qquad [\mathcal{L}_4, \mathcal{L}_3] = \mathcal{L}_2. \tag{6}$$

2.1 Integration of the field equations

The Einstein equations [1]

$$Ric_{\ \mu}^{\ \nu} = \lambda\,\delta_{\ \mu}^{\ \nu}$$

give 4 independent equations

$$\begin{aligned}
(I) \qquad & \frac{\ddot\beta}{\beta} + \frac{\dot\beta}{\beta}\left(2\frac{\dot\gamma}{\gamma} - \frac{\dot\alpha}{\alpha}\right) - \epsilon\frac{\alpha^2\beta^2}{2\gamma^4} + \epsilon\lambda\alpha^2 = 0,\\
(II) \qquad & \frac{\ddot\gamma}{\gamma} + \frac{\dot\gamma}{\gamma}\left(\frac{\dot\beta}{\beta} + \frac{\dot\gamma}{\gamma} - \frac{\dot\alpha}{\alpha}\right) + \epsilon\frac{\alpha^2\beta^2}{2\gamma^4} + \epsilon\,\lambda\,\alpha^2 = 0,\\
(III) \qquad & \frac{\ddot\beta}{\beta} + 2\frac{\ddot\gamma}{\gamma} - \frac{\dot\alpha}{\alpha}\left(\frac{\dot\beta}{\beta} + 2\frac{\dot\gamma}{\gamma}\right) + \epsilon\lambda\alpha^2 = 0.
\end{aligned}$$

The last relation, using the first two, simplifies to

$$(IV) \qquad 4\frac{\dot\beta\dot\gamma}{\beta\gamma} + 2\frac{\dot\gamma^2}{\gamma^2} + \epsilon\frac{\alpha^2\beta^2}{2\gamma^4} + 2\epsilon\lambda\alpha^2 = 0.$$

Subtracting (IV) to twice (II) we get

$$2\frac{\ddot\gamma}{\gamma} - 2\frac{\dot\gamma}{\gamma}\left(\frac{\dot\beta}{\beta} + \frac{\dot\alpha}{\alpha}\right) + \epsilon\frac{\alpha^2\beta^2}{2\gamma^4} = 0, \tag{7}$$

[1] In our notations the spheres have positive curvature.

which suggest to fix up the time coordinate arbitrariness by imposing $\alpha\,\beta = 1$. The previous relation decouples to the integrable equation

$$\frac{\ddot{\gamma}}{\gamma} + \frac{\epsilon}{4\gamma^4} = 0 \qquad \Longrightarrow \qquad \dot{\gamma}^2 - \frac{\epsilon}{4\gamma^2} = E. \tag{8}$$

Defining $\rho = \beta^2\gamma^2$, and combining (I) with (II) we get

$$\ddot{\rho} = -4\epsilon\lambda\,\gamma^2, \tag{9}$$

and relation (IV) reduces to

$$\frac{\dot{\rho}}{\rho}\frac{\dot{(\gamma^2)}}{\gamma^2} + \frac{\epsilon}{2\gamma^4} + 2\epsilon\lambda\,\frac{\gamma^2}{\rho} = 0. \tag{10}$$

So we need to integrate (8) for γ, then compute ρ and impose (10). Let us discuss separately the two signatures.

2.2 Minkowskian signature

In this case E cannot vanish. One gets

$$\gamma^2 = E\left(t^2 + \frac{1}{4E^2}\right), \qquad \rho = \rho_0 + mt + \lambda E\left(\frac{t^2}{2E^2} + \frac{t^4}{3}\right). \tag{11}$$

Imposing (10) we get $\rho_0 = -\dfrac{\lambda}{16E^3}$ and m remains free. In order to get rid of the factor E in λE it is sufficient to divide the metric by E. To compare to previous work let us define $4l^2E^2 = 1$. After obvious algebra we end up with

$$\left\{\begin{array}{l} g = 4l^2\dfrac{u}{c}\,\sigma_1^2 - \dfrac{c}{u}\,dt^2 + c(\sigma_2^2 + \sigma_3^2), \\[2ex] c = t^2 + l^2, \qquad u = mt + \lambda\left(-l^4 + 2l^2t^2 + \dfrac{t^4}{3}\right). \end{array}\right. \tag{12}$$

First obtained by Cahen and Defrise [3], see formula (13.48) with $e = k = 0$ in [17]. It has Petrov type D.

2.3 Euclidean signature

In this case $E = 0$ is possible. Let us first dispose with this case. Using $t = \gamma^2$ as a new variable we have $\rho = l + mt - \frac{2}{3}\lambda\,t^3$. This time (10) requires $m = 0$. So the metric can be written

$$g = \Delta\,\sigma_1^2 + \frac{dt^2}{\Delta} + t(\sigma_2^2 + \sigma_3^2), \qquad \Delta = \frac{l}{t} - \frac{2\lambda}{3}t^2. \tag{13}$$

This metric was obtained by Dancer and Strachan [9]. It is Kähler, with complex structure $J = dt \wedge \sigma_1 + t\,\sigma_2 \wedge \sigma_3$.

For $E \neq 0$ there are 2 cases, according to its sign. Since the derivations are rather similar to the minkowskian case, let us just state the results. For $E > 0$, with the same relation between l and E, we obtain

$$\left\{ \begin{array}{l} g = 4l^2 \dfrac{u}{c}\,\sigma_1^2 + \dfrac{c}{u}\,dt^2 + c(\sigma_2^2 + \sigma_3^2), \\[2ex] c = t^2 - l^2, \qquad u = mt + \lambda\left(l^4 + 2l^2t^2 - \dfrac{t^4}{3}\right). \end{array} \right. \tag{14}$$

The metric for $E < 0$ follows from (14) by the changes $c \to -c$ and $\lambda \to -\lambda$.

Let us first observe that the parameter l is not essential: one can get rid of it by the changes

$$g \to \frac{g}{l^4}, \qquad \tau = \frac{t}{l}, \qquad x \to \frac{x}{l^2}, \qquad y \to \frac{y}{l}, \qquad z \to \frac{z}{l}.$$

Therefore the metric displays two essential constants: m and λ which are expected for the general solution.

The metric (14) was first derived by Lorenz-Petzold in [11]. The Weyl tensor is

$$W^+ = \frac{3m + 8\lambda l^3}{6(t-l)^3}\,M, \qquad W^- = \frac{3m - 8\lambda l^3}{3(t+l)^3}\,M, \quad M = \mathrm{diag}\,(-2, 1, 1), \tag{15}$$

so there is room for metrics with self-dual Weyl tensor. Let us consider the case

$$W^+ = 0 \quad \to \quad u = -\frac{\lambda}{3}(t + 3l)(t-l)^3, \qquad t \geq l. \tag{16}$$

Positivity requires $\lambda < 0$, so that defining

$$\frac{s}{l} = \frac{t - 2b}{t + 2b}, \qquad -8b = \frac{3}{2|\lambda| l^2},$$

the metric becomes

$$g_{SD} = \frac{3}{2|\lambda|(t+2b)^2}\left(\frac{t+b}{t}\,\sigma_1^2 + \frac{t}{t+b}\,dt^2 + t(\sigma_2^2 + \sigma_3^2)\right), \tag{17}$$

which was derived in [22], and shown to be complete for $b = -1$.

2.4 Integrable geodesic flow

It is known [24], [8] that most Petrov type D vacuum metrics (the C metric being a notable exception) exhibit at least one Killing-Yano tensor, the square of which induces a Killing-Stäckel tensor (following the same terminology as in [22]) and this last one is essential to the integrability of the geodesic flow. However we suspect that this property could remain true for many type D Einstein metrics. Writing the metric

$$g = 4l^2\,\frac{u}{c}\,\sigma_1^2 + \epsilon\,\frac{c}{u}\,dt^2 + c(\sigma_2^2 + \sigma_3^2), \tag{18}$$

with

$$c = t^2 - \epsilon\, l^2, \qquad u = mt + \lambda\left(\epsilon l^4 + 2l^2t^2 - \epsilon\frac{t^3}{3}\right), \tag{19}$$

and taking the obvious tetrad, we found the following Killing-Yano tensor

$$Y = \epsilon\, l\, e^0 \wedge e^1 + t\, e^2 \wedge e^3, \tag{20}$$

the square of which produces the Killing-Stäckel tensor

$$S = c\Big((e^2)^2 + (e^3)^2\Big). \tag{21}$$

Taking for hamiltonian

$$2H = g^{ij}\,\Pi_i\,\Pi_j = \frac{1}{4l^2}\frac{c_\epsilon}{u_\epsilon}\,\Pi_x^2 + \epsilon\frac{u_\epsilon}{c_\epsilon}\,\Pi_t^2 + \frac{1}{c_\epsilon}\Big(\Pi_y^2 + (\Pi_z - y\Pi_x)^2\Big), \tag{22}$$

the Killing vectors induce observables linear in the momenta

$$\widetilde{\mathcal{L}}_1 = \Pi_x, \qquad \widetilde{\mathcal{L}}_2 = \Pi_y - z\Pi_x, \qquad \widetilde{\mathcal{L}}_3 = \Pi_z, \qquad \widetilde{\mathcal{L}}_4 = y\,\Pi_z - z\,\Pi_y - \frac{1}{2}(y^2 - z^2)\Pi_x, \tag{23}$$

which are conserved

$$\{H, \widetilde{\mathcal{L}}_i\} = 0,\ \ i = 1, \ldots 4, \tag{24}$$

while the Killing-Stäckel tensor induces a conserved observable which is quadratic in the momenta

$$\mathcal{S} = \Pi_y^2 + (\Pi_z - y\Pi_x)^2, \qquad \{H, \mathcal{S}\} = 0, \tag{25}$$

which is not reducible to a bilinear form with respect to the Killing vectors (23).

This dynamical system is therefore integrable since H, $\mathcal{S}$, Π_x, Π_z are in involution for the Poisson bracket. Writing the action as

$$S = Et + p\,x + q\,z + A(t), \qquad p = \Pi_x, \quad q = \Pi_z, \tag{26}$$

the Hamilton-Jacobi equation separates and we end up with

$$\epsilon\frac{u}{c}\left(\frac{dA}{dt}\right)^2 = \frac{c}{u}\frac{p^2}{4l^2} + \frac{\mathcal{S}}{c} - 2E. \tag{27}$$

3 Type III metrics

In this case the Lie algebra is defined as

$$[\mathcal{L}_1, \mathcal{L}_2] = 0, \qquad [\mathcal{L}_2, \mathcal{L}_3] = 0, \qquad [\mathcal{L}_3, \mathcal{L}_1] = \mathcal{L}_3. \tag{28}$$

A representation by differential operators is

$$\mathcal{L}_1 = \partial_x + z\,\partial_z, \qquad \mathcal{L}_2 = \partial_y, \qquad \mathcal{L}_3 = \partial_z, \tag{29}$$

and the invariant Maurer-Cartan 1-forms are

$$\sigma_1 = dx, \quad \sigma_2 = dy, \quad \sigma_3 = e^{-x}\,dz, \quad \Longrightarrow \quad d\sigma_1 = d\sigma_2 = 0, \quad d\sigma_3 = \sigma_3 \wedge \sigma_1. \tag{30}$$

We will look for diagonal metrics of the form

$$g = \beta^2\,\sigma_1^2 + \gamma^2\,\sigma_2^2 + \delta^2\,\sigma_3^2 + \epsilon\alpha^2\,dt^2. \tag{31}$$

If $\beta^2 = \delta^2$ the metric exhibits a fourth Killing vector

$$\mathcal{L}_4 = z\,\partial_x + \frac{1}{2}(z^2 - e^{2x})\partial_z, \tag{32}$$

and the algebra closes up to

$$[\mathcal{L}_1, \mathcal{L}_4] = \mathcal{L}_4, \qquad [\mathcal{L}_3, \mathcal{L}_4] = \mathcal{L}_1. \tag{33}$$

3.1 Flat space

For future use let us look for flat space within our cooordinates choice. An easy computation shows that it is given by

$$g_0 = \sigma_2^2 + t^2(\sigma_1^2 + \sigma_3^2) - dt^2 = dy^2 + t^2\,\frac{(dz^2 + dr^2)}{r^2} - dt^2, \qquad r = e^x. \tag{34}$$

The flattening coordinates are

$$x_1 = y, \qquad x_2 = \frac{tz}{r}, \qquad x_3 = \frac{t}{2r}(-1 + z^2 + r^2), \qquad \tau = \frac{t}{2r}(1 + z^2 + r^2),$$

which gives

$$g_0 = d\vec{r}\cdot d\vec{r} - d\tau^2, \qquad \vec{r} = (x_1, x_2, x_3).$$

3.2 Integration of the field equations

Following the same procedure as for the type II case, we obtain for independent equations

$$\begin{array}{ll}
(I) & \dfrac{\dot\delta}{\delta} = \dfrac{\dot\beta}{\beta}, \\[2ex]
(II) & \dfrac{\ddot\beta}{\beta} + \dfrac{\dot\beta}{\beta}\left(\dfrac{\dot\beta}{\beta} + \dfrac{\dot\gamma}{\gamma} - \dfrac{\dot\alpha}{\alpha}\right) + \epsilon\left(\dfrac{1}{\beta^2} + \lambda\right)\alpha^2 = 0, \\[2ex]
(III) & \dfrac{\ddot\gamma}{\gamma} + \dfrac{\dot\gamma}{\gamma}\left(2\dfrac{\dot\beta}{\beta} - \dfrac{\dot\alpha}{\alpha}\right) + \epsilon\,\lambda\,\alpha^2 = 0, \\[2ex]
(IV) & \dfrac{\dot\beta^2}{\beta^2} + 2\dfrac{\dot\beta\dot\gamma}{\beta\gamma} + \epsilon\left(\dfrac{1}{\beta^2} + \lambda\right)\alpha^2 = 0.
\end{array}$$

Relations (I) and (II)-(IV) integrate up to [2]

$$\dot\beta = c\,\alpha\,\gamma, \qquad c \in \mathbb{R}, \qquad \delta = \beta. \tag{35}$$

The time coordinate choice

$$\alpha = \frac{\beta}{\gamma} \qquad \Longrightarrow \qquad \delta = \beta = \beta_0\,e^{ct}.$$

To determine γ we have to use (III) which becomes

$$\frac{\ddot\gamma}{\gamma} + \frac{\dot\gamma^2}{\gamma^2} + c\,\frac{\dot\gamma}{\gamma} + \epsilon\lambda\beta_0^2\,\frac{e^{2ct}}{\gamma^2} = 0. \tag{36}$$

This equation does linearize in γ^2 to

$$\ddot{(\gamma^2)} + c\,\dot{(\gamma^2)} + 2\epsilon\lambda\beta_0^2\,e^{2ct} = 0, \tag{37}$$

and the remaining relation (IV) becomes

$$c\,\dot{(\gamma^2)} + c^2\,\gamma^2 + \epsilon(1 + \lambda\,\beta^2) = 0. \tag{38}$$

Let us organize the discussion according to the values of c.

[2] The coefficient between β and δ can be set to 1 by rescaling the coordinate z.

3.3 The metrics

We will consider first the special case $c = 0$. Relation (38) gives $\beta^2 = -1/\lambda$ and (37) is easily integrated to $\gamma^2 = \gamma_0 + \gamma_1 t + \epsilon t^2$. By a translation of t we can set $\gamma_1 \to 0$ and by a rescaling of z we can set $c_2 \to 1$, so we can write the metric

$$g = \frac{1}{|\lambda|}\Big[\sigma_1^2 + \sigma_3^2 + \gamma^2\,\sigma_2^2 + \epsilon\frac{dt^2}{\gamma^2}\Big], \qquad \gamma^2 = \gamma_0 + \epsilon t^2, \qquad \lambda < 0. \tag{39}$$

Let us emphasis that all the metrics will have negative Einstein constant.

For the minkowskian signature we must have $\gamma_0 > 0$. By a scaling of the variables y and t we can set $\gamma_0 = 1$. This leaves us with

$$g = \frac{1}{|\lambda|}\left\{dx^2 + e^{-2x}\,dz^2 + (1-t^2)\,dy^2 - \frac{dt^2}{(1-t^2)}\right\}. \tag{40}$$

The change of coordinates

$$\mu = \frac{1}{2}\Big[e^x + (1+z^2)\,e^{-x}\Big], \qquad \tan\phi = \frac{1}{2z}\Big[e^{2x} - (1-z^2)\Big], \tag{41}$$

leads to

$$g = \underbrace{\frac{1}{|\lambda|}\left\{\frac{d\mu^2}{\mu^2-1} + (\mu^2-1)\,d\phi^2\right\}}_{g_0} + \underbrace{\frac{1}{|\lambda|}\left\{(1-t^2)\,dy^2 - \frac{dt^2}{1-t^2}\right\}}_{g_1}, \tag{42}$$

on which we recognize a product of 2-dimensional Einstein metrics with the *same* scalar curvature: the euclidean $g_0 = \mathbb{H}^2$ and the lorentzian $g_1 = AdS_2$, so we end up with 6 Killing vectors. Let us notice that this it is a well known fact [1][p. 44] that for a product to be Einstein, it is mandatory that both two dimensional metrics in the product have the same Einstein constant.

For the euclidean signature, according to the sign of γ_0 we have 3 cases:

$$\begin{array}{lll}
\gamma_0 > 0 & g = \dfrac{1}{|\lambda|}\left\{\sigma_1^2 + \sigma_3^2 + \dfrac{1}{\cos^2\tau}\Big[\sigma_2^2 + d\tau^2\Big]\right\}, & \\
\gamma_0 = 0 & g = \dfrac{1}{|\lambda|}\left\{\sigma_1^2 + \sigma_3^2 + \dfrac{1}{\tau^2}\Big[\sigma_2^2 + d\tau^2\Big]\right\}, & \lambda < 0. \\
\gamma_0 < 0 & g = \dfrac{1}{|\lambda|}\left\{\sigma_1^2 + \sigma_3^2 + \dfrac{1}{\sinh^2\tau}\Big[\sigma_2^2 + d\tau^2\Big]\right\}, &
\end{array}$$

We have again decomposable Einstein metrics made up of two copies of $\mathbb{H}^2$.

Let us consider the more general case for which c does not vanish. We obtain

$$\tilde{f}(t) \equiv c_2^2\,\gamma^2 = -\epsilon + \gamma_1\,e^{-ct} - \frac{\epsilon\lambda\beta_0^2}{3}\,e^{2ct}. \tag{43}$$

Taking as variable $s = \beta_0\,e^{ct}$, and cleaning up the irrelevant parameters, we eventually obtain the Einstein metric

$$g = s^2(\sigma_1^2 + \sigma_3^2) + f(s)\,\sigma_2^2 + \epsilon\,\frac{ds^2}{f(s)}, \qquad f(s) = -\epsilon + \frac{\gamma_0}{s} - \frac{\epsilon\lambda}{3}\,s^2. \tag{44}$$

The metric exhibits the extra Killing vector (32) but is no longer decomposable. It contains two essential constants: γ_0 and λ which are expected for the general solution.

For $\epsilon = -1$ it was first obtained by Stewart and Ellis [15] and by Cahen and Defrise [3] and re-discovered later on in [14], [13] and more recently in [7]. In [17] the metric is given by formulas (13.9)and (13.48) in which one has to take $e = l = 0$ and $k = -1$. The Weyl tensor has a single non-vanishing component $\Psi_2 = -\dfrac{c}{2s^3}$ giving Petrov type D.

For $\epsilon = +1$, this metric was obtained by Lorenz-Petzold [11]. Using the obvious vierbein, we obtain for the Weyl tensor

$$W^+ = W^- = \frac{\gamma_0}{2s^3}\,M, \qquad\qquad M = \text{diag}(1, -2, 1), \tag{45}$$

giving Petrov type $(D^+,\, D^-)$.The Weyl tensor is self-dual if and only if $\gamma_0 = 0$.

In this case, for negative λ, we obtain a complete Einstein metric in the following way: let us change the variable s into $u = \sqrt{|\lambda|}\, s$. Then

$$f(s) \to h(u) = \frac{1}{3u}\,(u^3 - 3u + 2c), \qquad\qquad 2c = 3\sqrt{|\lambda|}\,\gamma_0,$$

so that the choice $c = -1$ gives a double root and for metric :

$$g = \frac{1}{|\lambda|}\left(u^2(\sigma_1^2 + \sigma_3^2) + |\lambda|\, h\, dy^2 + \frac{du^2}{h}\right), \qquad h(u) = \frac{(u-2)}{3u}(u+1)^2. \tag{46}$$

Positivity requires $u > 2$ and the metric becomes singular for $u = 2$. That this singularity is only apparent follows from a local analysis. If we take as new variables:

$$\xi \approx \sqrt{\frac{8}{3}(u-2)} \to 0, \qquad \tilde{y} = \frac{3}{4}\sqrt{|\lambda|}\, y,$$

the local form of the metric becomes a product metric $\mathbb{H}^2 \times \mathbb{R}^2$

$$g \approx \frac{1}{|\lambda|}\Big(4(\sigma_1^2 + \sigma_3^2) + \xi^2\, d\tilde{y}^2 + d\xi^2\Big), \qquad \tilde{y} \in\ [0, 2\pi].$$

The $u = 2$ singularity is therefore a removable polar-like singularity.

3.4 The special case $\gamma_0 = 0$

From (45) we see that for $\gamma_0 = 0$ the metric is conformally flat, so we must recover symmetric spaces with 10 Killing vectors instead of 4.

Let us first consider the minkowskian signature for $\lambda > 0$. We can write the metric

$$g_M^+ = \frac{3}{\lambda}\left[t^2(dx^2 + e^{-2x}\, dz^2) - \frac{dt^2}{1+t^2} + (1+t^2)\, du^2\right], \qquad t = \sqrt{\frac{\lambda}{3}}\, s, \quad u = \sqrt{\frac{\lambda}{3}}\, y. \tag{47}$$

The coordinates

$$\left\{\begin{array}{l} z^1 = tz\, e^{-x}, \quad z^2 = t(\sinh x + e^{-x}\, z^2/2), \quad z^3 = \sqrt{1+t^2}\, \cos u, \\[2ex] z^0 = t(\cosh x + e^{-x}\, z^2/2), \quad z^4 = \sqrt{1+t^2}\, \sin u, \end{array}\right. \tag{48}$$

are constrained by $(z^1)^2+(z^2)^2+(z^3)^2-(z^0)^2+(z^4)^2=1$ and

$$g_M^+=\frac{3}{\lambda}\Big((dz^1)^2+(dz^2)^2+(dz^3)^2-(dz^0)^2+(dz^4)^2\Big), \tag{49}$$

which is de Sitter metric, with isometry group enhanced to $O(4,1)$.

For $\lambda<0$ we start from

$$g_M^-=\frac{3}{|\lambda|}\left[t^2(dx^2+e^{-2x}\,dz^2)-\frac{dt^2}{1-t^2}+(1-t^2)\,du^2\right],\qquad t=\sqrt{\frac{|\lambda|}{3}}\,s,\qquad u=\sqrt{\frac{|\lambda|}{3}}\,y. \tag{50}$$

The coordinates

$$\left\{\begin{array}{l} z^1=tz\,e^{-x},\qquad z^2=t(\sinh x+e^{-x}\,z^2/2),\qquad z^3=\sqrt{1+t^2}\,\cosh u,\\[2mm] z^0=t(\cosh x+e^{-x}\,z^2/2),\qquad z^4=\sqrt{1+t^2}\,\sinh u,\end{array}\right. \tag{51}$$

are constrained by $(z^1)^2+(z^2)^2-(z^3)^2-(z^0)^2+(z^4)^2=1$ and

$$g_M^-=\frac{3}{\lambda}\Big((dz^1)^2+(dz^2)^2-(dz^3)^2-(dz^0)^2+(dz^4)^2\Big), \tag{52}$$

and the isometry group is enhanced to $O(3,2)$.

For the euclidean signature, positivity requires $\lambda<0$. We start from

$$g_E^-=\frac{3}{|\lambda|}\left[t^2(dx^2+e^{-2x}\,dz^2)-\frac{dt^2}{t^2-1}+(t^2-1)\,du^2\right],\qquad t=\sqrt{\frac{|\lambda|}{3}}\,s,\qquad u=\sqrt{\frac{|\lambda|}{3}}\,y. \tag{53}$$

The coordinates

$$\left\{\begin{array}{l} z^1=tz\,e^{-x},\qquad z^2=t(\sinh x+e^{-x}\,z^2/2),\qquad z^3=\sqrt{t^2-1}\,\cos u,\\[2mm] z^0=t(\cosh x+e^{-x}\,z^2/2),\qquad z^4=\sqrt{t^2-1}\,\sin u,\end{array}\right. \tag{54}$$

are constrained by $(z^1)^2+(z^2)^2+(z^3)^2-(z^0)^2+(z^4)^2=-1$ and

$$g_M^-=\frac{3}{\lambda}\Big((dz^1)^2+(dz^2)^2-(dz^3)^2-(dz^0)^2+(dz^4)^2\Big), \tag{55}$$

and the metric lies on the manifold $\mathbb{H}^4$.

The de Sitter metric (53) is written in quite “exotic” coordinates. Since this metric is of some importance, we give in appendix A, the explicit form of its Killing vectors.

3.5 Integrable geodesic flow

With the obvious tetrad, we found the following Killing-Yano and Killing-Stackel tensors

$$Y=s\,e^3\wedge e^1,\qquad\Rightarrow\qquad S=s^2\Big((e^1)^2+(e^3)^2\Big). \tag{56}$$

Let us consider the geodesic flow induced by the Hamiltonian

$$2H\equiv g^{ij}\,\Pi_i\,\Pi_j=\frac{1}{f(s)}\,\Pi_y^2+\frac{\Pi_x^2+e^{2x}\Pi_z^2}{s^2}+\epsilon\,f(s)\,\Pi_s^2. \tag{57}$$

The KS tensor S gives for conserved quantity

$$\mathcal{S}=\Pi_x^2+e^{2x}\Pi_z^2,\qquad\qquad \{H,\mathcal{S}\}=0. \tag{58}$$

It cannot be obtained from symmetrized tensor products of Killing vectors because their corresponding linear conserved quantities are

$$\widetilde{\mathcal{L}}_1=\Pi_x+z\,\Pi_z,\quad \widetilde{\mathcal{L}}_2=\Pi_y,\quad \widetilde{\mathcal{L}}_3=\Pi_z,\quad \widetilde{\mathcal{L}}_4=z\,\Pi_x+\frac{1}{2}\Big(z^2-e^{2x}\Big)\Pi_z,\quad \{H,\widetilde{\mathcal{L}}_i\}=0.$$

The dynamical system with hamiltonian H is therefore integrable, since it exhibits 4 independent conserved quantities: H, $\mathcal{S}$, Π_y, Π_z in involution for the Poisson bracket. Writing the action as

$$S=E\,t+p\,y+q\,z+A(s),\qquad\qquad p=\Pi_y,\quad q=\Pi_z \tag{59}$$

we get for separated Hamilton-Jacobi equation

$$\left(\frac{dA}{ds}\right)^2=\epsilon\left(\frac{2E}{f}-\frac{\mathcal{S}}{s^2\,f}-\frac{p^2}{f^2}\right). \tag{60}$$

Let us consider now the Bianchi V case.

4 Type V metrics

In this case the Lie algebra is

$$[\mathcal{L}_1,\mathcal{L}_2]=\mathcal{L}_2,\qquad [\mathcal{L}_2,\mathcal{L}_3]=0,\qquad [\mathcal{L}_3,\mathcal{L}_1]=-\mathcal{L}_3, \tag{61}$$

with the Killing vectors

$$\mathcal{L}_1=\partial_x-y\partial_y-z\partial_z,\qquad \mathcal{L}_2=\partial_y,\qquad \mathcal{L}_3=\partial_z, \tag{62}$$

and the invariant Maurer-Cartan 1-forms

$$\sigma_1=dx,\ \sigma_2=e^x\,dy,\ \sigma_3=e^x\,dz,\quad \Rightarrow\quad d\sigma_1=0,\quad d\sigma_2=\sigma_1\wedge\sigma_2,\quad d\sigma_3=\sigma_1\wedge\sigma_3. \tag{63}$$

We will look again for a diagonal metric of the form (31).

4.1 The flat space

Let us first determine the flat space Bianchi V metric. It is easy to check that it is given by

$$g_0=t^2(\sigma_1^2+\sigma_2^2+\sigma_3^2)-dt^2=t^2\,\gamma-dt^2, \tag{64}$$

where the metric γ is the Poincaré metric for $\mathbb{H}^3$:

$$\gamma\equiv\sigma_1^2+\sigma_2^2+\sigma_3^2=\frac{dy^2+dz^2+d\rho^2}{\rho^2},\qquad \rho=e^{-x},$$

which has 6 Killing vectors. The flattening coordinates for (64) are

$$x_1=\frac{ty}{\rho},\quad x_2=\frac{tz}{\rho},\quad x_3=\frac{t}{2\rho}(-1+y^2+z^2+\rho^2),\quad \tau=\frac{t}{2\rho}(1+y^2+z^2+\rho^2), \tag{65}$$

leading to

$$g_0\equiv t^2(\sigma_1^2+\sigma_2^2+\sigma_3^2)-dt^2=d\vec{r}\cdot d\vec{r}-d\tau^2,\qquad \vec{r}=(x_1,x_2,x_3). \tag{66}$$

4.2 Integration of the field equations

The independent equations are now

$$
\begin{array}{ll}
(I) & \dfrac{\ddot\beta}{\beta}+\dfrac{\dot\beta}{\beta}\left(\dfrac{\dot\gamma}{\gamma}+\dfrac{\dot\delta}{\delta}-\dfrac{\dot\alpha}{\alpha}\right)+\epsilon(2+\lambda\,\beta^2)\dfrac{\alpha^2}{\beta^2}=0,\\[2mm]
(II) & \dfrac{\ddot\gamma}{\gamma}+\dfrac{\dot\gamma}{\gamma}\left(\dfrac{\dot\beta}{\beta}+\dfrac{\dot\delta}{\delta}-\dfrac{\dot\alpha}{\alpha}\right)+\epsilon(2+\lambda\,\beta^2)\dfrac{\alpha^2}{\beta^2}=0,\\[2mm]
(III) & \dfrac{\ddot\delta}{\delta}+\dfrac{\dot\delta}{\delta}\left(\dfrac{\dot\beta}{\beta}+\dfrac{\dot\gamma}{\gamma}-\dfrac{\dot\alpha}{\alpha}\right)+\epsilon(2+\lambda\,\beta^2)\dfrac{\alpha^2}{\beta^2}=0,\\[2mm]
(IV) & \dfrac{\dot\beta\dot\gamma}{\beta\gamma}+\dfrac{\dot\gamma\dot\delta}{\gamma\delta}+\dfrac{\dot\beta\dot\delta}{\beta\delta}+\epsilon(3+\lambda\,\beta^2)\dfrac{\alpha^2}{\beta^2}=0,
\end{array}
\qquad (V)\quad \frac{\dot\delta}{\delta}-2\frac{\dot\beta}{\beta}+\frac{\dot\gamma}{\gamma}=0. \qquad (67)
$$

The differences (I)-(II) and (I)-(III) integrate to

$$
\frac{\dot\gamma}{\gamma}-\frac{\dot\beta}{\beta}=c\,\frac{\alpha}{\beta\gamma\delta},\qquad\qquad \frac{\dot\delta}{\delta}-\frac{\dot\beta}{\beta}=c_2\,\frac{\alpha}{\beta\gamma\delta},
$$

and (V) implies $c_2=-c$.

This suggests to fix up the time coordinate by imposing

$$
\alpha=\beta\gamma\delta\quad\Longrightarrow\quad \gamma=\gamma_0\,e^{ct}\,\beta,\qquad \delta=\delta_0\,e^{-ct}\,\beta,\qquad \alpha=\gamma_0\,\delta_0\,\beta^3.
$$

By a rescaling of the coordinates y and z, we may set $\gamma_0=\delta_0=1$ and relation (I) becomes

$$
D_t\left(\frac{\dot\beta}{\beta}\right)+\epsilon\,\beta^4(2+\lambda\,\beta^2)=0,\quad\Longrightarrow\quad \frac{\dot\beta^2}{\beta^2}+\epsilon\,\beta^4(1+\lambda\,\beta^2/3)=E.
$$

Eventually relation (IV) gives $E=c^2/3\geq 0$. Summarizing, we have obtained for the Einstein metric

$$
g=\beta^2\Big(\sigma_1^2+e^{2ct}\,\sigma_2^2+e^{-2ct}\,\sigma_3^2+\epsilon\;\beta^4\,dt^2\Big),\qquad \frac{\dot\beta^2}{\beta^2}=\frac{c^2}{3}-\epsilon\,\beta^4(1+\lambda\,\beta^2/3). \qquad (68)
$$

For the minkowskian signature, this result was first obtained by Schücking and Heckmann [16] and written, in [17][p. 192] as

$$
g=-d\tau^2+S^2(\tau)\Big(\sigma_1^2+F^{\sqrt3}\,\sigma_2^2+F^{-\sqrt3}\,\sigma_3^2\Big),
$$

with the relations

$$
3\left(\frac{dS}{d\tau}\right)^2=3+\frac{\Sigma^2}{S^4}+\lambda\,S^2,\qquad\qquad F^{\sqrt3}=\exp\left(2\Sigma\int\frac{d\tau}{S^3(\tau)}\right).
$$

Upon the identifications

$$
d\tau=\beta^3(t)\,dt,\qquad S(\tau)=\beta(t),\qquad \Sigma=c,
$$

the differential equation for $S(\tau)$ gives the differential equation for $\beta(t)$ and for F we get $F^{\sqrt3}=e^{2ct}$ showing full agreement with (68).

4.3 The special case $E = c = 0$

This special case leads to metrics with enhanced symmetries, namely non-compact symmetric spaces with 10 Killing vectors. Among these, as mentioned in [17], we expect de Sitter metrics.

The differential equation (68) becomes

$$dt = \frac{d\beta}{\beta^3 \sqrt{-\epsilon - \epsilon\lambda\beta^2/3}}. \tag{69}$$

Taking $\beta \to s$ as a new variable, we get the metric

$$g = s^2\Big(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\Big) - \frac{ds^2}{1 + \frac{\lambda\, s^2}{3}}. \tag{70}$$

The minkowskian or euclidean character of the metric does depend solely on the range taken by the variable t, and in the $\lambda \to 0$ limit we recover, as it should, the flat space metric (64).

For $\lambda > 0$, we can have only a minkowskian metric. As explained at the beginning of this section, we expect a higher symmetry. Defining $\sqrt{\dfrac{\lambda}{3}}\, s = \dfrac{2t}{1-t^2}$ we can write the metric:

$$g_M^+ = \frac{12}{\lambda}\frac{1}{(1-t^2)^2}\Big(t^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - dt^2\Big),$$

on which we recognize a symmetric space, since by using the flattening coordinates (65), we have

$$g_M^+ = \frac{12}{\lambda}\frac{d\vec{r}\cdot d\vec{r} - d\tau^2}{(1 + \vec{r}^{\,2} - \tau^2)^2}.$$

Indeed, using the constrained coordinates

$$z_0 = \frac{1 - \vec{r}^{\,2} + \tau^2}{1 + \vec{r}^{\,2} - \tau^2}, \qquad \vec{z} = \frac{2\vec{r}}{1 + \vec{r}^{\,2} - \tau^2}, \qquad z_4 = \frac{2\tau}{1 + \vec{r}^{\,2} - \tau^2}, \qquad z_0^2 + \vec{z}^{\,2} - z_4^2 = 1,$$

we see that we end up with de Sitter metric

$$g_M^+ = \frac{3}{\lambda}\Big(dz_0^2 + d\vec{z}\cdot d\vec{z} - dz_4^2\Big),$$

and the isometry group enlarges to $O(4,1)$. As for Bianchi type III, we get de Sitter metric in some “exotic” coordinates. Since this could be perhaps useful for other applications, we give in Appendix A the form of its Killing vectors.

For $\lambda < 0$ we have, for Minkowskian signature, anti de Sitter metric

$$g_M^- = \frac{12}{|\lambda|(1+t^2)^2}\Big(t^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - dt^2\Big).$$

For the euclidean signature we get

$$g_E^- = \frac{3}{|\lambda|}\Big[\mathrm{ch}^2\theta(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + d\theta^2\Big],$$

which is also a locally symmetric space. To give the embedding in $\mathbb{R}^5$ let us first define a set of 4 coordinates $(\vec{r},\, x^0)$ by

$$\vec{r} = \Big(e^x\, y,\ e^x\, z,\ -\sinh x + e^x(y^2+z^2)/2\Big), \qquad x^0 = \cosh x + e^x(y^2+z^2)/2,$$

which are constrained by $\vec{r}^{\,2} - (x^0)^2 = -1$. One can check that

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = dx^2 + e^{2x}(dy^2+dz^2) = d\vec{r}\cdot d\vec{r} - (dx^0)^2.$$

Then, defining the coordinates $(\vec{z},\, z^0,\, z^4)$ by

$$\vec{z} = \cosh\theta\,\vec{r}, \qquad z^0 = \cosh\theta, \qquad z^4 = \sinh\theta,$$

we conclude to

$$g_E^- = \frac{3}{|\lambda|}\Big(d\vec{z}\cdot d\vec{z} - (dz^0)^2 + (dz^5)^2\Big), \qquad \vec{z}^{\,2} - (z^0)^2 + (z^4)^2 = -1, \tag{71}$$

which shows that the metric g_E^- lives on the manifold $\mathbb{H}^4$.

4.4 The general case $E \neq 0$

In relation (68), let us introduce as a new variable

$$\rho = \frac{|c|}{\beta^2} > 0 \quad \Longrightarrow \quad \frac{\rho\, d\rho}{\sqrt{P(\rho)}} = \pm\frac{2c\, dt}{\sqrt{3}}, \qquad\qquad P(\rho) \equiv \rho(\rho^3 - 3\epsilon\,\rho - \epsilon\lambda|c|), \tag{72}$$

which gives for the metric

$$g = \frac{|c|}{\rho}\left(\sigma_1^2 + \gamma^2\,\sigma_2^2 + \frac{1}{\gamma^2}\,\sigma_3^2 + \frac{3}{4}\,\epsilon\,\frac{d\rho^2}{P(\rho)}\right), \qquad\qquad \gamma^2 \equiv e^{2|c|t}. \tag{73}$$

Remark: Due to the symmetric role played by $(\sigma_2,\, \sigma_3)$, the coefficients of σ_2^2 and of σ_3^2 may be interchanged and this corresponds to the exchange $(c \leftrightarrow -c)$ or $\left(\gamma \leftrightarrow \dfrac{1}{\gamma}\right)$. This means that if the metric (73) is Einstein, then

$$g = \frac{|c|}{\rho}\left(\sigma_1^2 + \frac{1}{\gamma^2}\,\sigma_2^2 + \gamma^2\,\sigma_3^2 + \frac{3}{4}\,\epsilon\,\frac{d\rho^2}{P(\rho)}\right), \tag{74}$$

will be Einstein too. We will use this observation to get rid of the sign in relation (72) and to take $c > 0$.

4.5 Minkowskian signature

In [16] the results are given up to the quadrature for $\beta(t)$. For the Einstein metric of interest this quadrature requires the use of elliptic functions. The technical details are given in the appendix; using these results we get the final form of the metrics, according to the sign of the Einstein constant.

We have to take $\epsilon = -1$. In this case the cubic polynomial $P(\rho) = \rho(\rho^3 + 3\rho + \lambda c)$ has, no matter what the value of c is, always 2 real and 2 complex conjugate roots (recall that we exclude $\lambda = 0$). So we fix $c = 1$ and, to express most conveniently the roots of P, we parametrize the Einstein constant according to

$$\lambda = 2\,\sinh(\theta), \qquad\qquad \theta \in \mathbb{R}\backslash\{0\}.$$

We will use now the results from appendix B to give the explicit form of the metric.

1. $\boxed{\text{For } \lambda < 0 :}$

In this case the roots are

$$a = -2\,\sinh(\theta/3)\ >\ b = 0, \qquad a_1 = \sqrt{3}\,\cosh(\theta/3), \quad b_1 = \sinh(\theta/3),$$

so we have

$$\left\{\begin{array}{l} A = \sqrt{3 + 12\,\sinh^2(\theta/3)}, \\[2mm] B = \sqrt{3 + 4\,\sinh^2(\theta/3)}, \end{array}\right. \qquad\qquad k^2 = \frac{(A+B)^2 - 4\,\sinh^2(\theta/3)}{4AB}$$

and

$$\mathrm{sn}\,v_0 = \sqrt{\frac{2B}{A + B - 2\,\sinh(\theta/3)}}.$$

In formula (73) we have to transform $d\rho$ into dv to get eventually

$$g_M = \frac{1}{\rho}\left(\sigma_1^2 + \gamma^2\,\sigma_2^2 + \frac{1}{\gamma^2}\,\sigma_3^2 - \frac{3}{AB}\,(dv)^2\right), \qquad\qquad v \in [0, v_0), \tag{75}$$

where ρ and γ^2 are given respectively by

$$\rho = \frac{aB\,\mathrm{cn}^2\,v}{B\,\mathrm{cn}^2\,v - A\,\mathrm{sn}^2\,v\,\mathrm{dn}^2\,v}, \tag{76}$$

and by

$$\gamma^2 = \left(e^{-\xi v}\frac{H(v_0+v)\,\Theta_1(v_0+v)}{H(v_0-v)\,\Theta_1(v_0-v)}\right)^{\sqrt{3}}, \qquad \xi = 2\left(\frac{\Theta'}{\Theta}(v_0) + \frac{H_1'}{H_1}(v_0)\right). \tag{77}$$

2. $\boxed{\text{For } \lambda > 0 :}$

In this case the roots are

$$a = 0\ >\ b = -2\,\sinh(\theta/3), \qquad a_1 = \sqrt{3}\,\cosh(\theta/3), \quad b_1 = \sinh(\theta/3),$$

so we have

$$\left\{\begin{array}{l} A = \sqrt{3 + 4\,\sinh^2(\theta/3)}, \\[2mm] B = \sqrt{3 + 12\,\sinh^2(\theta/3)}, \end{array}\right. \qquad\qquad k^2 = \frac{(A+B)^2 - 4\,\sinh^2(\theta/3)}{4AB}.$$

The parameter k^2 remains unchanged while A and B are interchanged and v_0 becomes

$$\mathrm{sn}\, v_0 = \sqrt{\frac{2B}{A+B+2\,\sinh(\theta/3)}}.$$

The metric is still given by (75), where now ρ and γ^2 are respectively

$$\rho = \frac{|b| A\, \mathrm{sn}^2 v\, \mathrm{dn}^2 v}{B\, \mathrm{cn}^2 v - A\, \mathrm{sn}^2 v\, \mathrm{dn}^2 v}, \tag{78}$$

and by

$$\gamma^2 = \left(e^{-\xi v} \frac{H(v_0+v)\,\Theta_1(v_0+v)}{H(v_0-v)\,\Theta_1(v_0-v)} \right)^{\sqrt{3}}, \qquad \xi = 2\left(\frac{|b|}{AB} + \frac{\Theta'}{\Theta}(v_0) + \frac{H_1'}{H_1}(v_0) \right). \tag{79}$$

Using the complex null-tetrad [3]

$$m = \frac{1}{\sqrt{2}}(e^{ct}\,\beta\,\sigma_2 + i e^{-ct}\,\beta\,\sigma_3), \qquad k = \frac{1}{\sqrt{2}}(\beta^3\, dt - \beta\,\sigma_1), \qquad l = \frac{1}{\sqrt{2}}(\beta^3\, dt + \beta\,\sigma_1), \tag{80}$$

and defining $\mu = 1/\beta^2$, one has to use the differential equation (68) which gives

$$\frac{\dot{\mu}^2}{4} = \frac{c^2}{3}\,\mu^2 + 1 + \frac{\lambda}{3}\,\frac{1}{\mu}. \tag{81}$$

Just using these informations one can check, computing the curvature, the Einstein property of this metric. For the Weyl tensor we have obtained

$$\Psi_0 = c\,\mu^2\Big(1 - \frac{\dot{\mu}}{2}\Big), \quad \Psi_2 = \frac{c^2}{3}\,\mu^3, \quad \Psi_4 = -c\,\mu^2\Big(1 + \frac{\dot{\mu}}{2}\Big), \qquad \Psi_1 = \Psi_3 = 0, \tag{82}$$

which establishes the Petrov type I of the metric.

4.6 Euclidean signature

In this case $P(\rho) = \rho(\rho^3 - 3\rho - \lambda c)$. It has two real roots for $\lambda c \in (-\infty, -2) \cup (+2, +\infty)$, four real roots for $\lambda c \in [-2, 0) \cup (0, +2]$ and a double root for $\lambda c = \pm 2$. Since the parameter c is free, we can collapse $(-\infty, 0) \cup (0, +\infty)$ to two points by taking $c = 2/|\lambda|$. Therefore in this case elliptic functions are no longer required!

We have to discuss two cases:

1. $\boxed{\lambda < 0 :}$

We have $P(\rho) = \rho(\rho+2)(\rho-1)^2$ and

$$\frac{2c}{\sqrt{3}}\, dt = \frac{\rho\, d\rho}{|\rho - 1|\sqrt{\rho(\rho+2)}}.$$

[3] We follow strictly the notations of [17].

The change of variable $\rho = \dfrac{2s^2}{3-s^2}$ simplifies to

$$2c\ dt = \frac{4s^2\,ds}{(1-s^2)(3-s^2)}.$$

We obtain

$$\gamma^2 \equiv e^{2ct} = \frac{1+s}{|1-s|}\left(\frac{\sqrt{3}-s}{\sqrt{3}+s}\right)^{\sqrt{3}}, \tag{83}$$

and the Einstein metric

$$g_E = \frac{(3-s^2)}{|\lambda|\,s^2}\left(\sigma_1^2 + \gamma^2\,\sigma_2^2 + \frac{1}{\gamma^2}\,\sigma_3^2 + \frac{ds^2}{(1-s^2)^2}\right). \tag{84}$$

In fact we have two different metrics, according to the interval taken for s: either $s \in (-1,1)$ or $s \in (1,\sqrt{3})$.

2. $\boxed{\lambda > 0:}$

We have $P(\rho) = \rho(\rho-2)(\rho+1)^2$ and

$$\frac{2c}{\sqrt{3}}\ dt = \frac{\rho\,d\rho}{(\rho+1)\sqrt{\rho(\rho+2)}}, \qquad \rho > 2.$$

The change of variable $\rho = \dfrac{2}{1-s^2}$ simplifies to

$$\frac{2c}{\sqrt{3}}\ dt = -\frac{4\,ds}{(1-s^2)(3-s^2)}, \qquad s \in (-1,+1).$$

Deleting the sign we obtain

$$\gamma^2 \equiv e^{2c\,t} = \frac{\sqrt{3}-s}{\sqrt{3}+s}\left(\frac{1+s}{1-s}\right)^{\sqrt{3}} \tag{85}$$

and the Einstein metric

$$g_E = \frac{(1-s^2)}{\lambda}\left[\sigma_1^2 + \gamma^2\,\sigma_2^2 + \frac{1}{\gamma^2}\,\sigma_3^2 + \frac{3\,ds^2}{(3-s^2)^2}\right]. \tag{86}$$

Some remarks are now in order:

1. We have checked, using the obvious vierbein, the vanishing of the matrix B and that $\mathrm{Tr}\,A = \lambda$, which proves the Einstein character of both metrics and computed the Weyl tensor: it has Petrov type (I^+, I^-) and is never self-dual.

2. It is interesting to compare with the results in [20] for the Bianchi type A Einstein metrics with self-dual Weyl tensor: except for Bianchi type II, they all involve Painlevé transcendents.

3. Let us observe that the difference in structure between the minkowskian and euclidean type V case is quite unusual. Indeed we have seen for the Bianchi type II and III metrics that the change in the signature brings rather small variance.

5 Conclusion

We have been giving very simple derivations of the "diagonal" Bianchi type II, III and V Einstein metrics. The first two exhibit an integrable geodesic flow, while the third one gives rise to new euclidean metrics which can be expressed in terms of elementary functions. Let us observe that there is little room for diagonal type B euclidean metrics with self-dual Weyl tensor: we have seen that the corresponding metrics are conformally flat, in agreement with [19]. A question of interest is to what extent one could work out the tri-axial type II metric or the more general Bianchi VI_h and Bianchi VII_h metrics, a rather difficult aim to say nothing of the non-diagonal ones!

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Appendix

A De Sitter metric re-visited

A.1 de Sitter from Bianchi type III

Let us consider the de Sitter metric (47) written as

$$g_M^+ = \frac{3}{\lambda}\left[t^2\,\frac{dz^2+dv^2}{v^2}+(1+t^2)du^2-\frac{dt^2}{1+t^2}\right], \qquad u=\sqrt{\frac{\lambda}{3}}\,y, \qquad v=e^x. \tag{87}$$

The four standard Killing vectors are now

$$K_1 = v\,\partial_v + z\,\partial_z, \qquad K_2=\partial_u, \qquad K_3=\partial_z, \qquad K_4 = zv\,\partial_v+\frac{(z^2-v^2)}{2}\,\partial_z. \tag{88}$$

The remaining ones appear by pairs

$$-\frac{\sin u}{vf}\,\partial_u+\frac{f\,\cos u}{v}(v\,\partial_v+t\,\partial_t), \qquad \frac{\cos u}{vf}\,\partial_u+\frac{f\,\sin u}{v}(v\,\partial_v+t\,\partial_t),$$

$$-\frac{z\sin u}{vf}\,\partial_u+\frac{f\cos u}{v}(zv\,\partial_v-v^2\,\partial_z+zt\,\partial_t), \qquad \frac{z\cos u}{vf}\,\partial_u+\frac{f\sin u}{v}(zv\,\partial_v-v^2\,\partial_z+zt\,\partial_t),$$

and

$$\left\{\begin{array}{l} \dfrac{(v^2+z^2)\sin u}{vf}\,\partial_u+\dfrac{f\cos u}{v}\Big((v^2-z^2)v\,\partial_v+2v^2z\,\partial_z-(v^2+z^2)t\,\partial_t\Big), \\[2ex] -\dfrac{(v^2+z^2)\cos u}{vf}\,\partial_u+\dfrac{f\sin u}{v}\Big((v^2-z^2)v\,\partial_v+2v^2z\,\partial_z-(v^2+z^2)t\,\partial_t\Big), \end{array}\right. \qquad f=\frac{\sqrt{1+t^2}}{t}.$$

Despite the simple form of the metric in these coordinates, the symmetries are somewhat wild.

A.2 de Sitter from Bianchi type V

We have shown that the metric

$$g = s^2\Big(\sigma_1^2+\sigma_2^2+\sigma_3^2\Big) - \frac{ds^2}{1+\frac{\lambda\, s^2}{3}}, \qquad \lambda>0, \tag{89}$$

is de Sitter. Taking $\sinh\theta = \sqrt{\frac{\lambda}{3}}\,s$ and $v=e^{-x}$ as new variables the metric becomes

$$g = \frac{3}{\lambda}\Big(\sinh^2\theta\,\frac{dy^2+dz^2+dv^2}{v^2} - d\theta^2\Big), \qquad \lambda>0. \tag{90}$$

The first 3 dimensional piece in the metric is $\mathbb{H}^3$ in Poincaré coordinates, so we have 2 sub-algebras:

$$\mathcal{A}_1 = \Big\{P_1,\,P_2,\,M_3\Big\}, \qquad \mathcal{A}_2 = \Big\{Q_1,\,Q_2,\,L_3\Big\}. \tag{91}$$

The first one is $e(2)$ (M_3 is a rotation)

$$P_1=\partial_y, \quad P_2=\partial_z, \quad M_3 = -z\,\partial_y + y\,\partial_z, \tag{92}$$

and the second one is $\tilde{e}(2)$ (L_3 is a dilatation)

$$\left\{\begin{aligned} Q_1 &= \tfrac12\Big(-y^2+z^2+v^2\Big)\partial_y - yz\,\partial_z - yv\,\partial_v,\\ Q_2 &= -zy\,\partial_y + \tfrac12\Big(y^2-z^2+v^2\Big)\partial_z - zv\,\partial_v,\end{aligned}\right. \qquad L_3 = -y\,\partial_y - z\,\partial_z - v\,\partial_v, \tag{93}$$

We need 4 extra Killing vectors to get the 10 dimensional $so(4,1)$ Lie algebra for de Sitter metric. They are given by

$$\begin{aligned} C_1 &= -\frac{1}{\tanh\theta}\,\partial_v - \frac{1}{v}\,\partial_\theta,\\ C_2 &= \frac{1}{\tanh\theta}(v\,\partial_y - y\,\partial_v) - \frac{y}{v}\,\partial_\theta, \qquad C_3 = \frac{1}{\tanh\theta}(v\,\partial_z - z\,\partial_v) - \frac{z}{v}\,\partial_\theta,\\ C_4 &= -\frac{v}{\tanh\theta}(y\,\partial_y + z\,\partial_z) + \frac{(y^2+z^2-v^2)}{2\tanh\theta}\,\partial_v + \frac{y^2+z^2+v^2}{2v}\,\partial_\theta. \end{aligned} \tag{94}$$

B Elliptic functions: some tools

There are plenty of books on elliptic function theory, but we used mainly the books by Byrd and Friedman [2] and by Whittaker and Watson [25]. We use Jacobi rather than Weierstrass notation for elliptic functions. Similarly we use earlier Jacobi notation for the theta functions which is best adapted to our purposes. They are related to the more symmetric notations used in [25] according to

$$H(v)=\theta_1(w), \quad H_1(v)=\theta_2(w), \quad \Theta_1(v)=\theta_3(w), \quad \Theta(v)=\theta_4(w), \qquad w=\frac{\pi v}{2K}.$$

Let us start from the relation (72)

$$\frac{2dt}{\sqrt{3}} = \frac{\rho\,d\rho}{\sqrt{P(\rho)}}. \tag{95}$$

If the quartic polynomial $P(\rho)$ has 2 real roots, and therefore two complex conjugate ones, we will write it

$$P(\rho) = (\rho - a)(\rho - b)[(\rho - b_1)^2 + a_1^2], \qquad a > b.$$

In this case, the positivity of ρ and $P(\rho)$ requires $\rho \geq a$. One defines

$$A = \sqrt{(a - b_1)^2 + a_1^2} \quad > \quad B = \sqrt{(b - b_1)^2 + a_1^2}, \qquad k^2 = \frac{(A+B)^2 - (a-b)^2}{4AB} < 1,$$

where k^2 will be the parameter of the elliptic functions involved. Let us define the change of variable

$$\mathrm{sn}^2\, v = \frac{2B(\rho - a)}{D_+}, \qquad \mathrm{cn}^2\, v = \frac{D_-}{D_+}, \qquad \mathrm{dn}^2\, v = \frac{D_-}{2A(\rho - b)}, \tag{96}$$

with

$$D_\pm = A(\rho - b) \pm B(\rho - a) + (a - b)\sqrt{(\rho - b_1)^2 + a_1^2}, \tag{97}$$

and the parameters

$$s_0 \equiv \mathrm{sn}\, v_0 = \sqrt{\frac{2B}{A + B + a - b}} < 1, \quad s_1 \equiv \mathrm{sn}\, v_1 = \sqrt{\frac{2B}{A + B - a + b}} > 1,$$

for which the reader can check that $v_1 = K + iK' + v_0$.

The change of variable (96) transforms $\rho \in [a, +\infty)$ into $v \in [0, v_0) \subset [0, K_0)$. The inverse relation is [4]

$$\rho = \frac{aB\, c^2 - bA\, s^2 d^2}{B\, c^2 - A\, s^2 d^2}. \tag{98}$$

Using

$$\frac{\rho - a}{a - b} = \frac{A\, s^2 d^2}{B\, c^2 - A\, s^2 d^2}, \qquad \frac{\rho - b}{a - b} = \frac{B\, c^2}{B\, c^2 - A\, s^2 d^2},$$

$$\sqrt{(\rho - b_1)^2 + a_1^2} = AB\, \frac{d^2 - c^2 + c^2 d^2}{B\, c^2 - A\, s^2 d^2},$$

straightforward computations give

$$\frac{d\rho}{\sqrt{P(\rho)}} = \frac{2}{\sqrt{AB}}\, dv.$$

It remains to give the explicit form of $\gamma^2 = e^{2t}$ as a function of v by integrating (95), which becomes now:

$$\frac{2dt}{\sqrt{3}} = \frac{2}{\sqrt{AB}} \frac{aB\, c^2 - bA\, s^2\, d^2}{B\, c^2 - A\, s^2\, d^2}\, dv. \tag{99}$$

The relation

$$\frac{c_0^2}{s^2 - s_0^2} = -\frac{c_0}{2s_0 d_0}\left(\frac{H'}{H}(v_0 - v) + \frac{H'}{H}(v_0 + v) - 2\frac{\Theta'}{\Theta}(v_0)\right),$$

[4] From now on we will use the simplified notations $s \equiv \mathrm{sn}\,(v, k^2)$, $c \equiv \mathrm{cn}\,(v, k^2)$, $d \equiv \mathrm{dn}\,(v, k^2)$ as well as $s_0 = \mathrm{sn}\, v_0$, $s_1 = \mathrm{sn}\, v_1$ etc...

and a similar one, obtained by the substitution $v_0 \to v_1 = K + iK' + v_0$:

$$\frac{c_1^2}{s^2 - s_1^2} = \frac{c_0}{2s_0 d_0}\left(\frac{\Theta_1'}{\Theta_1}(v_0 - v) + \frac{\Theta_1'}{\Theta_1}(v_0 + v) - 2\frac{H_1'}{H_1}(v_0)\right),$$

allow us to integrate up to

$$\gamma^2 \equiv e^{2t} = \left(e^{-\xi v}\frac{H(v_0+v)\,\Theta_1(v_0+v)}{H(v_0-v)\,\Theta_1(v_0-v)}\right)^{\sqrt{3}}, \qquad \xi = 2\left(-\frac{b}{\sqrt{AB}} + \frac{\Theta'}{\Theta}(v_0) + \frac{H_1'}{H_1}(v_0)\right). \tag{100}$$

As the reader may notice, in [2][p. 135] a different change of variables is given, which differs from ours. It is

$$\operatorname{cn} u = \frac{(A-B)\rho - bA + aB}{(A+B)\rho - bA - aB}.$$

As a consequence we get in the metric (73) the term

$$-\frac{3}{4}\frac{d\rho^2}{P(\rho)} = -\frac{3}{AB}\left(\frac{du}{2}\right)^2.$$

To avoid the 1/2 factor we have used a duplication transformation to switch to our variable by $u = 2v$, having in mind that in the limit $\lambda \to 0$ we have $3/AB \to 1$.

C Curvature computations

Taking the obvious vierbein e_A, we define the connection ω and its self-dual components by

$$de_A + \omega_{AB} \wedge e_B = 0, \quad A = 0,1,2,3 \qquad \omega_a^{\pm} = \omega_{0a} \pm \frac{1}{2}\,\epsilon_{abc}\,\omega_{bc}, \quad a,b,c = 1,2,3. \tag{101}$$

The self-dual components of the curvature follow from

$$R_a^+ = d\omega^+ - \frac{1}{2}\,\epsilon_{abc}\,\omega_b^+ \wedge \omega_c^+, \qquad R_a^- = d\omega^- + \frac{1}{2}\,\epsilon_{abc}\,\omega_b^- \wedge \omega_c^-. \tag{102}$$

Using the 2-forms

$$\lambda_a^{\pm} = e_0 \wedge e_a \pm \frac{1}{2}\,\epsilon_{abc}\,e_b \wedge e_c,$$

the curvature can be expressed in terms of a triplet of 3×3 matrices

$$\begin{pmatrix} R_a^+ \\ R_a^- \end{pmatrix} = \begin{pmatrix} A_{ab} & B_{ab} \\ {}^t B_{ab} & C_{ab} \end{pmatrix}\begin{pmatrix} \lambda_b^+ \\ \lambda_b^- \end{pmatrix}, \qquad {}^t A = A, \qquad {}^t C = C. \tag{103}$$

Notice that the self-dual components of the Weyl tensor, defined by

$$W_a^+ = W_{ab}^+\,\lambda_a^+, \qquad W_a^- = W_{ab}^-\,\lambda_a^-,$$

are

$$W^+ = A - \frac{\operatorname{tr} A}{3}\,\mathbb{I}, \qquad W^- = C - \frac{\operatorname{tr} C}{3}\,\mathbb{I}. \tag{104}$$

For the diagonal Bianchi type V metrics considered here the matrices $W^{\pm}$ have the general structure

$$W^{\pm} = \begin{pmatrix} w_{11} & 0 & 0 \\ 0 & w_{22} & \pm w_{23} \\ 0 & \pm w_{23} & w_{33} \end{pmatrix}. \tag{105}$$

For the case $\lambda > 0$ we get:

$$\begin{cases} w_{11} = \dfrac{8\lambda}{3(1-s^2)^3}, \qquad w_{23} = -\dfrac{2\lambda}{(1-s^2)^2}, \\ w_{22} = \dfrac{2\lambda}{3}\dfrac{(\sqrt{3}s^3 - 3\sqrt{3}s - 2)}{(1-s^2)^3}, \\ w_{33} = -\dfrac{2\lambda}{3}\dfrac{(\sqrt{3}s^3 - 3\sqrt{3}s + 2)}{(1-s^2)^3}, \end{cases} \tag{106}$$

The eigenvalues are all different hence we have a "Petrov-like" type of the form (I^+, I^-). The conclusions are the same for the case $\lambda < 0$ for which we have

$$\begin{cases} w_{11} = \dfrac{-8\lambda s^6}{(3-s^2)^3}, \qquad w_{23} = \dfrac{2\lambda s^4}{(3-s^2)^2}, \\ w_{22} = \dfrac{2\lambda}{3}\dfrac{s^3(2s^3 - 9s^2 + 9)}{(3-s^2)^3}, \\ w_{33} = \dfrac{2\lambda}{3}\dfrac{s^3(2s^3 + 9s^2 - 9)}{(3-s^2)^3}. \end{cases} \tag{107}$$

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